Comments on the presymplectic formalism and the theory of regular Lagrangians with constraints

1995 J. Phys. A: Math. Gen. 28 L91

(http://iopscience.iop.org/0305-4470/28/3/006)

J. Phys. A: Math. Gen. 28 (1995) L91–L97. Printed in the UK

LETTER TO THE EDITOR

Comments on the presymplectic formalism and the theory of regular Lagrangians with constraints

José F Cariñena and Manuel F Rañada

Departamento de Física Teórica, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain

Received 4 November 1994

Abstract. The Lagrangian formalism for systems with constraints is developed using a singular Lagrangian defined in the extended tangent bundle $T(Q \times \mathbb{R})$. The dynamics defined by the new extended Lagrangian, that incorporates the constraints, is studied using the formalism of the presymplectic geometry. A comparative study with other geometric approaches is presented.

The theory of constrained systems includes the study of practical problems of great importance some of which are related to control theory, but the study of these systems is also important for it poses many questions which are intimately connected with some of the geometric methods used in modern theoretical mechanics (see [1–5] and references therein).

The standard method for incorporating constraint functions into equations of motion is the use of the so-called Lagrange multipliers. The relation between this method and the Lagrangian formalism can be studied using two different approaches.

(i) The original regular Lagrangian L is the appropriate Lagrangian but the presence of constraints introduce a perturbing effect in the free Euler–Lagrange equations that can be identified with the addition of a vertical non-Lagrangian perturbation.

(ii) The presence of Lagrange multipliers is related to the existence of a Lagrangian $\mathbb{L} \neq L$ defined in an extended space, but this new Lagrangian is singular.

Recently [4], approach (i) has been studied using the geometric tools from Lagrangian tangent bundle geometry. The purpose of this letter is to present a study of approach (ii) using the formalism of presymplectic geometry.

Suppose that a Lagrangian L is given. Then one can construct a semibasic 1-form θ_L (the associated Cartan form), an exact 2-form ω_L and an energy function E_L by

$$\theta_L = S^*(\mathrm{d}L) \qquad \omega_L = -\mathrm{d}\theta_L \qquad E_L = \Delta(L) - L$$

where S is the vertical endomorphism and Δ the Liouville vector field:

$$S = \frac{\partial}{\partial v^i} \otimes \mathrm{d}q^i \qquad \Delta = v^i \frac{\partial}{\partial v^i}.$$

The dynamics is represented by the flow of the vector field X_L solution of the equation

$$i(X_L)\omega_L = \mathrm{d}E_L.$$

If the 2-form ω_L is symplectic the Lagrangian L is regular; otherwise L is singular. If L is regular the solution X_L of this equation is uniquely determined and it turns out to be a second-order differential equation (hereafter shortened to SODE) vector field, i.e. $S(X_L) = \Delta$, and its integral curves satisfy the Euler–Lagrange equations. In coordinates X_L takes the form

$$X_L = v^i \frac{\partial}{\partial q^i} + f_L^i(q, v)\frac{\partial}{\partial v^i}$$

where $f_L^i(q, v)$ are the Lagrangian forces.

Next, we summarize some of the main characteristics of the theory of singular Lagrangians (for a review see [6, 7]).

(i) The Lagrangian L is said to be singular when the 2-form ω_L is not symplectic. If the rank of this 2-form is constant then the ω_L is called presymplectic.

(ii) Because $\mathrm{Ker}\,\omega_L \neq 0$ the dynamical equation is ill defined. The first condition to be satisfied is that the energy E_L must be projectable by $\mathrm{Ker}\,\omega_L$. This property leads to a submanifold M_1 in which the dynamical equation can be studied.

(iii) The geometrical algorithm for obtaining a submanifold C in which the dynamical equation admits a tangent solution was developed by Gotay *et al* [8–10]. The algorithm generates a decreasing sequence $\{M_k\}$ of submanifolds and then C is the limit of such a sequence (provided it exists). The restricted equation to C has solutions tangent to C.

(iv) In some cases the solution obtained in C is given by a non-SODE vector field. The conditions for the existence of a solution which is the restriction of a SODE can lead to a smaller final submanifold (in the Lagrangian formalism the dynamics must always be represented by SODE vector fields).

Suppose that our system is described by a regular Lagrangian function L on TQ (TQ is the velocity phase space of the configuration space Q) but it is subjected to a constraint force expressed by the presence of a constraint equation of the form

$$\phi(q^i, v^i) = 0.$$

We introduce a new configuration space $\mathcal{Q}$ of the form $\mathcal{Q} = Q \times \mathbb{R}$, the coordinates on the tangent bundle $T\mathcal{Q}$ being denoted $(q^i, \lambda, v^i, \zeta)$, and a new function $\mathbb{L}$ defined on $T\mathcal{Q}$ by

$$\mathbb{L}(q^i, \lambda, v^i, \zeta) = L(q^i, v^i) + \lambda\phi(q^i, v^i)$$

ϕ being the given constraint function on TQ.

The new Lagrangian $\mathbb{L}$ is singular because it does not depend on the velocity ζ of the coordinate λ. As stated above, we are interested in the case of $\omega_{\mathbb{L}}$ of constant rank. The following proposition relates this property with the form of ϕ.

Proposition 1. Let ω_ϕ denote the 2-form $\omega_\phi = -\mathrm{d}\theta_\phi$, $\theta_\phi = S^*(\mathrm{d}\phi)$. If $i(X)\omega_\phi = 0$ for any vertical field $X^v \in \mathcal{X}^v(TQ)$ then the extended Lagrangian 2-form $\omega_{\mathbb{L}}$ is presymplectic.

Proof. The 1-form $\theta_{\mathbb{L}}$ takes the form

$$\theta_{\mathbb{L}} = \theta_L + \lambda\theta_\phi$$

so that the expression of $\omega_{\mathbb{L}}$ in local coordinates is

$$\omega_{\mathbb{L}} = \left(\frac{\partial^2 L}{\partial q^j \partial v^i} + \lambda\frac{\partial^2 \phi}{\partial q^j \partial v^i}\right) \mathrm{d}q^i \wedge \mathrm{d}q^j + \mathbf{W}_{ji}\, \mathrm{d}q^i \wedge \mathrm{d}v^j + \left(\frac{\partial \phi}{\partial v^i}\right) \mathrm{d}q^i \wedge \mathrm{d}\lambda$$

where $\mathbf{W}_{ji}$ denotes the n-dimensional Hessian matrix of $\mathbb{L}$

$$\mathbf{W}_{ji} = W_{ji} + \lambda W_{ji}^{\phi} \qquad W_{ji} = \frac{\partial^2 L}{\partial v^j \partial v^i} \qquad W_{ji}^{\phi} = \frac{\partial^2 \phi}{\partial v^j \partial v^i}.$$

The expression obtained for $\mathbf{W}_{ji}$ can be interpreted as a pencil of quadratic forms. Therefore, when λ coincides with one of its eigenvalues the rank of $\omega_{\mathbb{L}}$ will be less than $2n$. If $i(X)\omega_{\phi} = 0$ for any vertical field $X^v \in \mathcal{X}^v(TQ)$, then W_{ji}^{ϕ} vanishes. Consequently, if ω_{ϕ} satisfies this property then $\omega_{\mathbb{L}}$ will have a constant rank $2n$ on TQ.

This proposition restricts the extended formalism to the case of affine constraints. In geometric terms, this is equivalent to the existence of Q of a function $h \in C^{\infty}(Q)$ and a 1-form $\alpha \in \Lambda^1(Q)$ such that ϕ takes form

$$\phi = \tilde{h} + \hat{\alpha}$$

where $\tilde{h}$ is the pull-back of h through the tangent bundle projection, $\tilde{h} = \tau_Q^* h$, and $\hat{\alpha} \in C^{\infty}(TQ)$ denotes a function linear on the fibres defined by $\hat{\alpha}(q, v) = \langle \alpha(q), v \rangle$. The situation $\phi = \tilde{h}$ corresponds to the so-called holonomic constraints and $\phi = \hat{\alpha}$ to non-holonomic constraints of the linear velocity-dependent type. The case $\alpha \in B^1(Q)$ means that the velocity-dependent function $\phi = \hat{\alpha}$ is the 'time derivative' of a holonomic constraint.

It was proved in [11, 12] that if L is a singular Lagrangian then the kernel of ω_L satisfies the following dimensional relation

$$\dim(\operatorname{Ker} \omega_L) \leqslant 2 \dim[V(\operatorname{Ker} \omega_L)].$$

Singular Lagrangians satisfying the equality, i.e. $\dim(\operatorname{Ker}\omega_L) = 2\dim[V(\operatorname{Ker}\omega_L)]$, are called type II. They are endowed with the following interesting characteristics.

(i) The action of the vertical endomorphism S on the kernel gives the vertical part, i.e. $S(\operatorname{Ker}\omega_L) = V(\operatorname{Ker}\omega_L)$.

(ii) If L is a type-II Lagrangian which admits a global dynamics, then there always exists a SODE field solution of the dynamics; if there is no global dynamics then this property is also true but the SODE solution need not be tangent to the constraint submanifold.

(iii) Assuming that S passes to the quotient under $\operatorname{Ker}\omega_L$ and that there is a global dynamics, its projection $\tilde{S}$ defines an integrable almost tangent structure if and only if L is of type II.

The expressions for $\omega_{\mathbb{L}}$ and $E_{\mathbb{L}}$ are

$$\omega_{\mathbb{L}} = \omega_L + \tilde{\alpha} \wedge \mathrm{d}\lambda - \lambda \widetilde{\mathrm{d}\alpha}$$

and

$$E_{\mathbb{L}} = E_L - \lambda \tilde{h}$$

because the $\hat{\alpha}$ term does not contribute to the energy function $E_{\mathbb{L}}$, for $\Delta(\hat{\alpha}) - \hat{\alpha} = 0$.

The kernel of $\omega_{\mathbb{L}}$, defined by

$$\operatorname{Ker}\omega_{\mathbb{L}} = \{Z \in \mathcal{X}(TQ) \mid i(Z)\omega_{\mathbb{L}} = 0\}$$

is a two-dimensional subbundle of TQ, the coordinate expression of two generators being given by

$$Z_1 = \frac{\partial}{\partial \lambda} - W^{rs}\alpha_s \frac{\partial}{\partial v^r} \qquad Z_2 = \frac{\partial}{\partial \zeta}$$

where $\alpha_r = \alpha_r(q)$ denote the components of the 1-form α, and W^{ij} is the inverse matrix of the Hessian W_{ij} which, as stated above, is assumed to be of maximal rank. Consequently $\operatorname{Ker}\omega_{\mathbb{L}}$ satisfies

$$\dim \operatorname{Ker}\omega_{\mathbb{L}} = 2\dim[V(\operatorname{Ker}\omega_{\mathbb{L}})].$$

Thus, if the constraint ϕ satisfies the assumption of proposition 1, then the extended Lagrangian $\mathbb{L}$ is of type II. □

Once the kernel of $\omega_{\mathbb{L}}$ has been obtained, we must study whether the reduced space $TQ/\operatorname{Ker}\omega_{\mathbb{L}}$ inherits the integrable almost tangent structure. Cantrijn *et al* [12] have proved that for type-II Lagrangians the test of this property consists of checking whether the image of $\mathcal{L}_Z S$ lies in $\operatorname{Ker}\omega_{\mathbb{L}}$ for any Z in $\operatorname{Ker}\omega_{\mathbb{L}}$ ($\mathcal{L}_Z S$ denotes the Lie derivative of S under Z). The following proposition studies the case of standard Lagrangians, that is, of regular Lagrangians defined by a pair (g, V) where g is a Riemannian metric on Q and V is a potential function (the co-called Lagrangians of mechanical type).

Proposition 2. Let the configuration space Q be a Riemannian manifold and the Lagrangian $L: TQ \to \mathbb{R}$ be defined by $L(q, v) = \frac{1}{2}g(v, v) - V(q)$ where $V \in C^\infty(Q)$. Then, if the constraint ϕ takes the form $\phi = \tilde{h} + \hat{\alpha}$, the (extended) vertical endormorphism field S defined on $T(Q \times \mathbb{R})$ passes to the quotient under $\operatorname{Ker}\omega_{\mathbb{L}}$.

Proof. In coordinates we have the following expression for S,

$$S = \frac{\partial}{\partial v^i} \otimes \mathrm{d}q^i + \frac{\partial}{\partial \zeta} \otimes \mathrm{d}\lambda$$

and, if α is a 1-form on Q, the Lie derivatives are

$$\mathcal{L}_{Z_1} S = \frac{\partial(W^{rs}\alpha_s)}{\partial v^i} \frac{\partial}{\partial v^r} \otimes \mathrm{d}q^i \qquad \mathcal{L}_{Z_2} S = 0.$$

Therefore, the condition $\operatorname{Im}(\mathcal{L}_Z S) \subset \operatorname{Ker}\omega_{\mathbb{L}}$ leads to

$$\frac{\partial}{\partial v^i}(W^{rs}\alpha_s) = 0.$$

If L is of mechanical type we have $W^{rs} = g^{rs}$, hence the condition is satisfied and the (extended) vertical endomorphism S passes to the quotient. □

Note that this result can be extended directly to mechanical Lagrangians with magnetic terms or to the more general case of g being a pseudo-Riemannian metric.

Since the extended Lagrangian $\mathbb{L}$ is of type II, if S passes to the quotient, then it defines an integrable almost tangent structure S' on $TQ/\operatorname{Ker}\omega_{\mathbb{L}}$. Thus it makes sense to look for second-order fields in this quotient space. The manifold $TQ/\operatorname{Ker}\omega_{\mathbb{L}}$ is $2n$-dimensional with induced coordinates given by $(q^i, w^i; i, = 1, \ldots, n)$ where $w^i = v^i + \lambda W^{ir}\alpha_r$. Obviously for holonomic constraints of the $\phi = \tilde{h}$ type, the w^is corresponding to the v^is and S' can be identified with the vertical endomorphism of TQ. This is not the case when $\phi = \hat{\alpha}$ where the quotient manifold cannot naturally be identified with TQ.

The extended Lagrangian $\mathbb{L}$ does not admit a global dynamics in TQ but the dynamics is restricted to $M_1 = \Phi^{-1}(0) \subset TQ$ where the constraint function Φ is given by the action of the non-vertical vectors of $\operatorname{Ker} \omega_{\mathbb{L}}$ on the energy function $E_{\mathbb{L}}$, i.e. $\Phi = Z_l(E_{\mathbb{L}})$.

We obtain

$$\Phi = Z_1(E_{\mathbb{L}}) = -\tilde{h} - W^{rs}\alpha_s \frac{\partial}{\partial v^r}\left(v^k \frac{\partial L}{\partial v^k} - L\right) = -\tilde{h} - \hat{\alpha}.$$

Consequently, the constraint equation $\phi(q^i, v^i) = 0$ that in the usual formalism must be used as a supplement of the equations of motion arises in this geometric approach in a natural way as the function defining the primary constraint submanifold.

The vector field $X_{\mathbb{L}}$, defined in TQ and satisfying the dynamical equation

$$i(X_{\mathbb{L}})\omega_{\mathbb{L}} = \mathrm{d}E_{\mathbb{L}}$$

on the primary constraint submanifold M_1, has the form

$$X_{\mathbb{L}} = v^i \frac{\partial}{\partial q^i} + \zeta \frac{\partial}{\partial \lambda} + F^i(q, v, \lambda, \zeta) \frac{\partial}{\partial v^i} + G(q, v, \lambda, \zeta) \frac{\partial}{\partial \zeta}$$

where G is an arbitrary function and the F^is are given by

$$F^i = f^i_L(q, v) + \lambda W^{ij}\left[\left(\frac{\partial \alpha_k}{\partial q^j} - \frac{\partial \alpha_j}{\partial q^k}\right) v^k + \frac{\partial h}{\partial q^j}\right] - \zeta W^{ij}\alpha_j$$

where $f^i_L(q, v)$ are the Lagrangian forces arising from L.

According to the geometric algorithm of Gotay *et al* [8–10], the primary constraint ϕ must be consistent with the dynamical equation, thus

$$\chi \equiv X_{\mathbb{L}}(\phi) = 0.$$

This new function χ represents a secondary constraint and it determines a new submanifold M_2 of M_1 by $M_2 = \chi^{-1}(0) \subset M_1$.

The following three particular situations are considered.

(i) $\phi = \tilde{h}, h \in C^\infty(Q)$. The function χ takes the form $\chi = h'_i v^i$, $h'_i = \partial h/\partial q^i$. A new constraint function is defined by $\psi \equiv X_{\mathbb{L}}(\chi)$ and the equation $\psi = 0$ determines the submanifold $M_3 \subset M_2$ by fixing the value of the coordinate λ as a function of q^i and v^i. We obtain

$$\lambda^*(q, v) = -\frac{1}{h'^2}(h''_{rs} v^r v^s + h'_r f^r_L)$$

where h'^2 denotes $h'^2 = W^{ij} h'_i h'_j$. This corresponds to the determination of the value of the Lagrange multiplier in the usual non-geometrical approach. Hence, the dynamical vector field $X_{\mathbb{L}}$ defined in TQ and tangent to M_3 takes the form

$$X_{\mathbb{L}} = v^i \frac{\partial}{\partial q^i} + \zeta \frac{\partial}{\partial \lambda} + \left\{ f^i_L(q, v) - \frac{1}{h'^2}(h''_{rs} v^r v^s + h'_r f^r_L) W^{ik} h'_k \right\} \frac{\partial}{\partial v^i} + G \frac{\partial}{\partial \zeta}.$$

The final submanifold $C = M_4 \subset M_3$ is defined by the equation $\psi' \equiv X_{\mathbb{L}}(\psi) = 0$ which can be written as $\zeta = \zeta(q, v)$. Finally the condition of tangency on C uniquely determines the function G, the field $X_{\mathbb{L}}$ is projectable to TQ and the equations of the projected integral curves are

$$\frac{\mathrm{d}}{\mathrm{d}t} q^i = v^i$$
$$\frac{\mathrm{d}}{\mathrm{d}t} v^i = f_L^i(q, v) - \frac{1}{h'^2}(h''_{rs} v^r v^s + h'_r f_L^r) W^{ik} h'_k.$$

We notice that these equations agree with the results obtained (using a different approach) in [4].

(ii) $\phi = \hat{\alpha}$, $\alpha = \mathrm{d}h$, $h \in C^\infty(Q)$. The equation $\chi = 0$, that determines the submanifold M_2, introduces a relation between the velocity ζ and q^i and v^i. We obtain

$$\zeta^*(q, v) = \frac{1}{h'^2}(h''_{rs} v^r v^s + h'_r f_L^r).$$

Notice that in this case we determine ζ without previously fixing the value of λ.

The (extended) dynamics is given by the restriction to $C = M_2$ of the vector field $X_{\mathbb{L}}$ that takes the form

$$X_{\mathbb{L}} = v^i \frac{\partial}{\partial q^i} + \zeta^* \frac{\partial}{\partial \lambda} + \left\{ f_L^i(q, v) - \frac{1}{h'^2}(h''_{rs} v^r v^s + h'_r f_L^r) W^{ik} h'_k \right\} \frac{\partial}{\partial v^i} + G \frac{\partial}{\partial \zeta}.$$

As in (i) the condition of tangency on C uniquely determines G as a function $G = G(q, v)$. The field $X_{\mathbb{L}}$ is projectable to TQ and the equations of the projected integral curves are

$$\frac{\mathrm{d}}{\mathrm{d}t} q^i = v^i$$
$$\frac{\mathrm{d}}{\mathrm{d}t} v^i = f_L^i(q, v) - \frac{1}{h'^2}(h''_{rs} v^r v^s + h'_r f_L^r) W^{ik} h'_k.$$

This result can also be obtained as follows: if $\phi = \hat{\alpha}$, $\alpha = \mathrm{d}h$, then the Lagrangian $\mathbb{L}$ can be written as

$$\mathbb{L} = L - \zeta h + \frac{\mathrm{d}}{\mathrm{d}t}(\lambda h).$$

The third term on the right is a gauge term, and the second term is similar to the one in case (i) but with $-\zeta$ instead of λ.

(iii) $\phi = \hat{\alpha}$, $\alpha \in \Lambda^1(Q)$. The equation $\chi = 0$, that determines directly M_2 as the final constraint submanifold, introduces a relation between the velocity ζ and q^i and v^i and λ. We obtain

$$\zeta^*(q, v, \lambda) = \frac{1}{\alpha^2} \left(X_L(\hat{\alpha}) + \lambda W^{ij} \left(\frac{\partial \alpha_k}{\partial q^j} - \frac{\partial \alpha_j}{\partial q^k} \right) v^k \alpha_i \right)$$

where $X_L(\hat{\alpha}) = (\partial \alpha_r / \partial q^s) v^r v^s + \alpha_r f_L^r$ and $\alpha^2 = W^{rs} \alpha_r \alpha_s$. Finally the (extended) dynamics is given by the restriction to $C = M_2$ of the vector field $X_{\mathbb{L}}$ but, in this case, the coordinate λ is not determined as a function of q and v, and, because of this, $X_{\mathbb{L}}$ is not projectable to TQ.

To summarize, we have proved that only affine functions can be incorporated into the Lagrangian mechanics by means of Lagrange multipliers and that for these functions the extended Lagrangian $\mathbb{L}$, although singular, is endowed with the 'semi-regular' properties of type-II Lagrangians. When the constraints are holonomic this presymplectic approach agrees with the standard way of reducing the configuration space and the same is true for constraints given by exact forms (integrable constraints), but for the velocity-dependent case the extended coordinate λ remains coupled to the physical degrees of freedom. Finally, in recent years the theory of Lagrange multipliers enlarging the configuration space of a system has received much attention in connection with the Faddeev–Jackiw approach [13–15] to quantization of linear Lagrangians [16]. Thus we think that this matter is particularly interesting and that the use of the symplectic formalism for constrained systems still requires much development.

Partial financial support by DGICYT (Madrid), grant PB93-0582, is acknowledged.

References

[1] Weber R W 1983 *Modern Developments in Analytical Mechanics (Proc. IUTAM Symp.)* ed S Benenti, M Francaviglia and A Lichnerowicz (Torino: Accademia)
[2] Weber R W 1985 *Arch. Rat. Mech. Anal.* **91** 309
[3] Koiller J 1992 *Arch. Rat. Mech. Anal.* **118** 113
[4] Cariñena J F and Rañada M F 1993 *J. Phys. A: Math. Gen.* **26** 1335
[5] Rañada M F 1994 *J. Math. Phys.* **35** 748
[6] Sundermeyer K 1982 *Constrained Dynamics (Lecture Notes in Physics 169)* (Berlin: Springer)
[7] Cariñena J F 1990 Theory of singular Lagrangians *Forschr. Phys.* **38** 641
[8] Gotay M J, Nester J M and Hinds G 1978 *J. Math. Phys.* **19** 2388
[9] Gotay M J and Nester J M 1979 *Ann. Inst. H. Poincaré* A **30** 129
[10] Gotay M J and Nester J M 1980 *Ann. Inst. H. Poincaré* A **32** 1
[11] Cariñena J F and Ibort L A 1985 *J. Phys. A: Math. Gen.* **18** 3335
[12] Cantrijn F, Cariñena J F, Crampin M and Ibort L A 1986 *J. Geom. Phys.* **3** 353
[13] Jackiw R 1994 Constrained quantization without tears *Constraint Theory and Quantization Methods* ed F Colomo, L Lusanna and G Marmo (Singapore: World Scientific)
[14] Barcelos-Neto J and Wotzasek C 1992 *Mod. Phys.* A **7** 4981
[15] Barcelos-Neto J and Braga N R 1994 *J. Math. Phys.* **35** 3497
[16] Cariñena J F, López C and Rañada M F 1988 *J. Math. Phys.* **29** 1134